UNIT 1
EXPONENTS AND LOGARITHMS

AFM

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1.1 FUNCTIONS

- **Objective:** I will be able to identify a relation as a function, evaluate a function for given values, graph a function, and determine the domain and range of a function.

- **Vocabulary**

Function	Relation	Domain	Range	Linear Function
Independent Variable	Dependent Variable	Evaluating a Function	Graph of a Function	Zeros of a Function
Vertical Line Test	Y-intercept			

1.1 FUNCTIONS

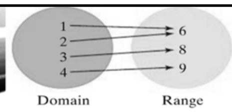


FIGURE 2.4(a) (repeated)

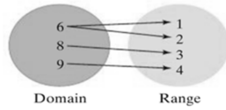


FIGURE 2.4(b) (repeated)

- Relations
  - A relation is any set of ordered pairs.
    - The set of all first components is called the domain (x-values)
    - The set of all second components is called the range (y-values)

- Functions
  - A relation in which each member of the domain corresponds to exactly one member of the range is a function.
    - A function cannot have one x-value with two or more y-values
    - A function can have two different x-values with the same y-value.

1.1 FUNCTIONS

- Functions as Equations
  - Functions are usually given as equations rather than sets of ordered pairs.
    - Ex: $y = 0.012x^2 - 0.2x + 8.7$
  - In the example above, x, is the independent variable because it can be assigned any value from the domain.
  - The dependent variable is y because its value depends on x.
  - NOTE: Not all equations with the variables x and y define functions. Each x-value must produce only one y-value for the equation to represent a function.

1.1 FUNCTIONS

- Function Notation
  - When an equation represents a function, the function is often named by a letter such as $f, g, h, F, G, or\ H.$ (NOTE: any letter can be used.)
  - The special notation $f(x)$
    - Read as "f of x" or "f at x"
    - x is the input and $f(x)$ is the output value of the function
    - Represents the value of the function at the number x
    - Ex: $y = 0.012x^2 - 0.2x + 8.7$ becomes $f(x) = 0.012x^2 - 0.2x + 8.7$

1.1 FUNCTIONS

- When a number is input for x, it replaces x in the special notation (Ex: $f(30)$).
  - In this case, use 30 as the input and determine the output $f(30)$.

$$f(30) = 0.012(30)^2 - 0.2(30) + 8.7$$
$$f(30) = 10.8 - 6 + 8.7$$
$$f(30) = 13.5$$

1.1 FUNCTIONS

- Graphs of Functions
  - The graph of a function is the graph of its ordered pairs.
  - To graph
    - Choose x values.
    - Compute $f(x)$ by evaluating $f(x)$ at x.
    - Form the ordered pairs.
    - Plot the values.
  - All functions with equations of the form $f(x) = mx + b$ graph straight lines and are called linear functions.

1.1 FUNCTIONS

- Vertical Line Test
  - Not all graphs in the regular coordinate system represent functions.
  - Can use the vertical line test to determine if a graph is a function.
    - Values of x paired with two or more different values of y form a vertical line.
    - If any vertical line intersects a graph in more than one point, the graph does not define y as a function of x.

1.1 FUNCTIONS

- Obtaining Information from a Graph
  - At the right or left of a graph you will find closed dots, open dots, or arrows.
    - Closed dots – the graph does not extend beyond this point, and the point belongs to the graph.
    - Open dots – the graph does not extend beyond this point, and the point does NOT belong to the graph.
    - Arrows – the graph extends indefinitely in the direction in which the arrow points.

1.1 FUNCTIONS

- Identifying Domain and Range from a Graph
  - Interval Notation
    - Square brackets indicate endpoints that are included in an interval.
    - Parentheses indicate endpoints that are not included in an interval
    - Parentheses are always used with ∞ or $-\infty$
  - Examples: Set-Builder Notation vs. Interval Notation
    - Set-Builder: $\{x| -4 \leq x \leq 2\}$
    - Interval Notation: $[-4,2]$

1.1 FUNCTIONS

- Identifying Intercepts
  - The zeros of a function f are the x-values for which $f(x) = 0.$ (x-intercepts).
  - To find the y-intercept, find the point at which the graph crosses the y-axis ($f(0)$).
  - A function can have more than one x-intercept but at most one y-intercept.

1.2 EXPONENTIAL FUNCTIONS

- Objective
  - I will be able to evaluate and graph exponential functions. I will be able to recognize the natural base e and use it in an exponential function.

- Vocabulary

Exponential Function	Base	Natural Base	Natural Exponential Function	Asymptote

1.2 EXPONENTIAL FUNCTIONS

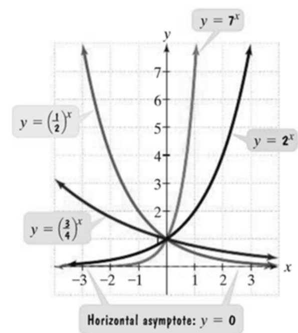
- Evaluate Exponential Functions
  - Functions whose equations contain a variable in the exponent are called exponential functions.
  - The exponential function *f* with base b is defined by
    - $f(x)=b^x$ or $y=b^x$ where x is any real number
    - b is a positive constant other than one ($b>0$ and $b\neq 1$) called the base
  - Use a calculator to evaluate exponential functions.

1.2 EXPONENTIAL FUNCTIONS

- Graphing Exponential Functions
  - Recall that $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
  - In these cases, when the function is in the form b^x, x must be rational, causing holes at irrational domain values.
    - The function $f(x)=b^x$ can be graphed at all values of x, meaning the graph will be continuous.

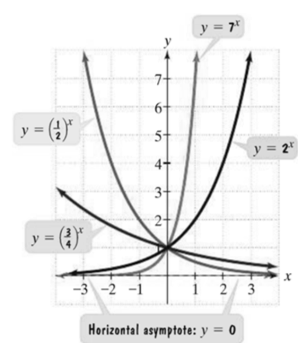
1.2 EXPONENTIAL FUNCTIONS

- Graphing
  - Choose values of x and evaluate for *f(x)*.
  - Plot the points and connect in a smooth curve.
  - Note that graphs of exponential functions have a horizontal asymptote (a line they approach but never touch or cross).



1.2 EXPONENTIAL FUNCTIONS

- Characteristics of $f(x)=b^x$
  - Domain: $(-\infty,\infty)$; Range: $(0,\infty)$
  - y-intercept: (0, 1); no x-intercepts
  - One-to-one function that has an inverse
  - Horizontal asymptote: $y=0$ (x-axis)



1.2 EXPONENTIAL FUNCTIONS

- Transformations: $f(x)=ab^{c(x-h)}+k$

Transformation	Equation	Description
Vertical Translation	$f(x)=b^x+k$	• $k>0$ shift upward k units • $k<0$ shift downward k units
Horizontal Translation	$f(x)=ab^{(x-h)}$	• $h>0$ $(x-h)$ shift right h units • $h<0$ $(x+h)$ shift left h units
Reflection	$f(x)=ab^{cx}$	• negative a reflects over x-axis • negative c reflects over y-axis
Vertical stretch/shrink	$f(x)=ab^x$	• $a>1$ vertically stretches graph • $0<a<1$ vertically shrinks graph
Horizontal stretch/shrink	$f(x)=b^{cx}$	• $c>1$ horizontally shrinks graph • $0<c<1$ horizontally stretches graph

1.2 EXPONENTIAL FUNCTIONS

- The Natural Base e
  - Defined as the value that $(1+1/n)^n$ approaches as n approaches infinity.
  - $e \approx 2.718281827$ or $e \approx 2.72$ when rounded to the nearest hundredth
  - The function $f(x)=e^x$ is called the natural exponential function.

1.3 COMPOUND INTEREST

- Objective
  - I will be able to use compound interest formulas.

- Vocabulary

Compound Interest	Principal	Compounded Semiannually	Compounded Quarterly	Continuous Compounding

1.3 COMPOUND INTEREST

- Compound Interest
  - Interest computed on your original investment as well as on any accumulated interest.
  - The initial amount invested is called the principal, P.
  - The annual percentage rate of interest, r, is compounded once per year.

1.3 COMPOUND INTEREST

- For one year, the total amount in such an account can be represented by

$$A= P+Pr = P(1+r)$$

- The formula for compound interest over time is

$$A = P(1+r)^t$$

1.3 COMPOUND INTEREST

- Interest can be compounded multiple times per year. When interest is compounded n times a year, we say that there are n compounding periods per year.

Name	**# Compounding periods/yr**	**Length of Each Period**
Semiannual Compounding	n = 2	6 months
Quarterly Compounding	n = 4	3 months
Monthly Compounding	n = 12	1 month
Daily Compounding	n = 365	1 day

1.3 COMPOUND INTEREST

- The above formula can be adjusted for the number of compounding periods in a year.

$$A=P(1+r/n)^{nt}$$

- This formula gives the total amount with a principal investment, P, compounded n times per year at an annual rate of r over t years.

1.3 COMPOUND INTEREST

- Continuous Compounding
  - Compound interest where the number of compounding periods increases infinitely.
  - The formula for compound interest is

$$A=Pe^{rt}$$

1.4 LOGARITHMIC FUNCTIONS

- Objective
  - I will be able to identify logarithmic functions and their properties. I will be able to convert between logarithmic and exponential forms.

- Vocabulary

Logarithmic Function	Logarithm	Exponential Form	Inverse Properties of Logarithms	

1.4 LOGARITHMIC FUNCTIONS

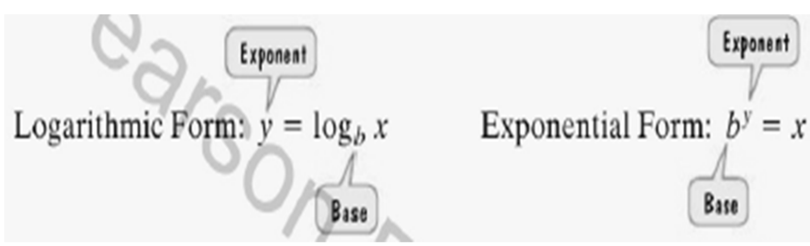
- Inverse Functions
  - Only one-to-one functions have inverses that are functions.
    - Pass the horizontal line test
    - Inverse found by switching x and y, then solve for y
  - If $f(a)=b$ then $f^{-1}(b)=a$ (switch the domain and range)
  - $f(f^{-1}(x))=x$ and $f^{-1}(f(x))=x$
  - Graph of f^{-1} is the reflection of f about the line $y=x$

1.4 LOGARITHMIC FUNCTIONS

- Logarithmic Functions
  - The inverse of $y=b^x$ is $x=b^y$
    - New notation needed to solve for y, called logarithmic notation
    - The inverse function of the exponential function with base b is called the logarithmic function with base b.
    - $y=\log_b x$ is equivalent to $b^y=x$ for $x>0$ and $b>0$, $b\neq 1$
      - $y=\log_b x$ is called logarithmic form
      - $b^y=x$ is called exponential form

1.4 LOGARITHMIC FUNCTIONS

- Knowing where the base and exponent are in each form will allow you to convert between the forms.



1.4 LOGARITHMIC FUNCTIONS

- Evaluating Logarithms
  - Remembering that logarithms are exponents makes it possible to evaluate some logarithms by inspection.
  - Ex: Evaluate $log_2 32$
    - Ask: 2 to what power gives 32?

$$2^5=32$$

$$log_2 32=5$$

1.4 LOGARITHMIC FUNCTIONS

- Basic Logarithmic Properties
  - Properties Involving 1
    - $logb_b=1$ because 1 is the exponent to which b must be raised to obtain b ($b^1=b$)
    - $log_b 1=0$ because 0 is the exponent to which b must be raised to obtain 1 ($b^0=1$)
  - Inverse Properties of Logarithms (for $b>0$ and $b\neq 1$)
    - $log_b b^x = x$
    - $b^{log_b x} = x$

1.4 LOGARITHMIC FUNCTIONS

- Common Logarithms
  - Logarithmic function with base 10 is called the common logarithmic function
  - Usually expressed as *f(x)*=logx.
  - Uses LOG key on calculator
  - Properties:

Properties of Common Logarithms

General Properties		Common Logarithms
1. $\log_b 1 = 0$		1. $\log 1 = 0$
2. $\log_b b = 1$		2. $\log 10 = 1$
3. $\log_b b^x = x$	Inverse properties	3. $\log 10^x = x$
4. $b^{\log_b x} = x$		4. $10^{\log x} = x$

1.4 LOGARITHMIC FUNCTIONS

- Natural Logarithms
  - Logarithmic function with base *e* is called the natural logarithmic function.
  - Usually expressed as *f(x)*=lnx
  - Uses LN key on calculator
  - Properties:

Properties of Natural Logarithms

General Properties		Natural Logarithms
1. $\log_b 1 = 0$		1. $\ln 1 = 0$
2. $\log_b b = 1$		2. $\ln e = 1$
3. $\log_b b^x = x$	Inverse properties	3. $\ln e^x = x$
4. $b^{\log_b x} = x$		4. $e^{\ln x} = x$

1.5 LOGARITHMIC GRAPHS

- Objective: I will be able to graph a logarithmic function including any transformations, and I will be able to determine its domain.

- Vocabulary:

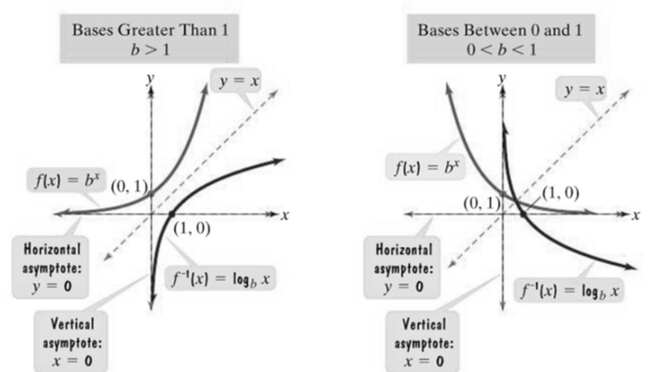
o Domain of Logarithmic Function		

1.5 LOGARITHMIC GRAPHS

- Graphs of Logarithmic Functions
  - Use the fact that logarithmic functions are the inverse of exponential functions
  - Recall that the graph of a logarithmic function is the reflection of its inverse over the line y = x.

1.5 LOGARITHMIC GRAPHS

- The base will determine the shape of the graph:



1.5 LOGARITHMIC GRAPHS

- Characteristics of Logarithmic Functions in the form $f(x)=\log_b x$
  - Domain: (0, ∞) ; Range: (-∞, ∞)
  - Graphs of all logarithmic parent functions pass through (1, 0)
  - There is no y-intercept
  - Asymptote: x = 0

1.5 LOGARITHMIC GRAPHS

- Transformations of $f(x)=a \log_b c(x-h)+k$

Transformation	**Equation**	**Description**
Vertical Translation	$f(x)=\log_b x+k$	• $k>0$ shift upward k units • $k<0$ shift downward k units
Horizontal Translation	$f(x)=\log_b(x–h)$	• $h>0$ $(x-h)$ shift right h units (asymptote: x = c) • $h<0$ $(x+h)$ shift left h units (asymptote: x = -c)
Reflection	$f(x)=a \log_b cx$	• negative a reflects over x-axis • negative c reflects over y-axis
Vertical stretch/shrink	$f(x)=a \log_b x$	• $a>1$ vertically stretches graph • $0<a<1$ vertically shrinks graph
Horizontal stretch/shrink	$f(x)=\log_b cx$	• $c>1$ horizontally shrinks graph • $0<c<1$ horizontally stretches graph

1.6 PROPERTIES OF LOGS

- *Objective*: I will be able to expand and/or condense logarithmic expressions using the product, quotient, and power rules. I will be able to convert from one base to another using the change-of-base property.

- Vocabulary:

Product Rule	Quotient Rule	Power Rule	Expanding a Logarithmic Expression	
Condense a Logarithmic Expression				

1.6 PROPERTIES OF LOGS

- The Product Rule
  - Recall that when multiplying the same base, exponents are added: $b^m * b^n = b^{m+n}$
  - Since logarithms are exponents, the product rule can be applied:

$$\log_b MN = \log_b M + \log_b N$$

  - b, M & N are positive real numbers; $b \neq 1$
  - When the product rule is applied, it is called expanding a logarithmic expression.

1.6 PROPERTIES OF LOGS

- The Quotient Rule
  - Recall that when dividing the same base, exponents are subtracted: $b^m / b^n = b^{m-n}$
  - Since logarithms are exponents, the quotient rule can be applied:

$$\log_b M/N = \log_b M - \log_b N$$

  - b, M & N are positive real numbers; $b \neq 1$

1.6 PROPERTIES OF LOGS

- The Power Rule
  - Recall that when raising an exponential expression to a power, multiply exponents: $(b^m)^n=b^{mn}$
  - Since logarithms are exponents, the power rule can be applied:

$$log_b M^p = p\ log_b M$$

  - b & M are positive real numbers; $b \neq 1$; p is any real number
  - Note that the graph of the expanded form may have a different domain than the original.

1.6 PROPERTIES OF LOGS

- Expanding & Condensing Logarithmic Expressions
  - Sometimes, more than one rule may need to be used when expanding or condensing a logarithmic expression.
  - Condensing a logarithmic function uses the reverse of the above properties.
  - Coefficients of logarithms must be 1 before you can condense them using the product and quotient rules.

1.6 PROPERTIES OF LOGS

- Change-of-Base Property
  - For any logarithmic bases a and b, and any positive number M,

$$log_b M = log_a M \; / \; log_a b$$

  - Base b is the original base while base a is the new base being introduced.
  - This property can be used with common logarithms and natural logarithms, as well.

1.7 EQUATIONS

- Objective - I will be able to solve exponential and logarithmic equations. I will be able to apply solving these equations to practical applications.

- Vocabulary

Exponential Equation	Logarithmic Equation	

1.7 EQUATIONS

- Exponential Equations
  - An exponential equation is an equation containing a variable in an exponent.
  - Some exponential equations can be solved by expressing each side as a power of the same base.
    - If b > 0 and $b \neq 1$, and $b^M = b^N$, then M = N.

1.7 EQUATIONS

- Most cannot be written so that each side has the same base.
  - Use logarithms!
  - If M = N, then $log_b M = log_b N$
  - If the exponential equations involves base 10, use common logarithms.
  - If it involves any other base, use natural logarithms.

1.7 EQUATIONS

- Using the Definition of a Logarithm to Solve Logarithmic Equations
  - A logarithmic equation is an equation containing a variable in a logarithmic expression.
  - Some logarithmic equations can be expressed in the form $log_b M = c$
    - Re-write these in exponential form ($b^c = M$) and solve for the variable.
    - Check proposed solutions. Include only values for which M > 0.
    - Note: x can still be a negative number so long as M > 0.

1.7 EQUATIONS

- Some logarithmic equations can be expressed in the form $log_b M = log_b N$
  - Use the one-to-one property to write without logarithms (M = N) and solve for the variable.
  - Check proposed solutions. Include only values for which M > 0 and N > 0.

1.8 GROWTH AND DECAY

- Objective - I will be able to model exponential growth and decay. I will be able to use logistic growth models for limited growth applications, and I will be able to re-write an exponential equation in the natural base.

- Vocabulary

Exponential Growth	Exponential Decay	Half-Life	Correlation Effect	

1.8 GROWTH AND DECAY

- Model Exponential Growth and Decay
  - With exponential growth or decay, quantities grow or decay at a rate directly proportional to their size.

1.8 GROWTH AND DECAY

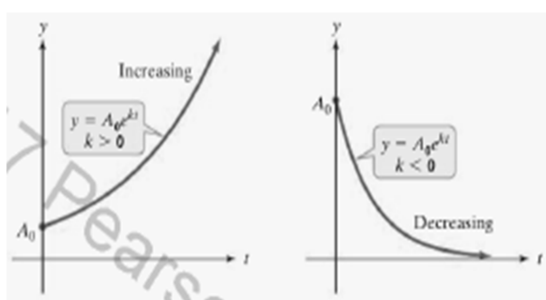
- Given by the model:

$$f(t) = A_0 e^{kt} \quad \text{or} \quad A = A_0 e^{kt}$$

  - If k > 0, the function models the amount, or size, of a growing entity.
    - A_0 is the original amount of the growing entity at time t = 0
    - A is the amount at time t
    - k is a constant representing growth rate
    - Ex: population growth

1.8 GROWTH AND DECAY

  - If k < 0, the function models the amount, or size, of a decaying entity.
    - k is a constant representing decay rate
    - Ex: half-life



(a) Exponential growth (b) Exponential decay

1.8 GROWTH AND DECAY

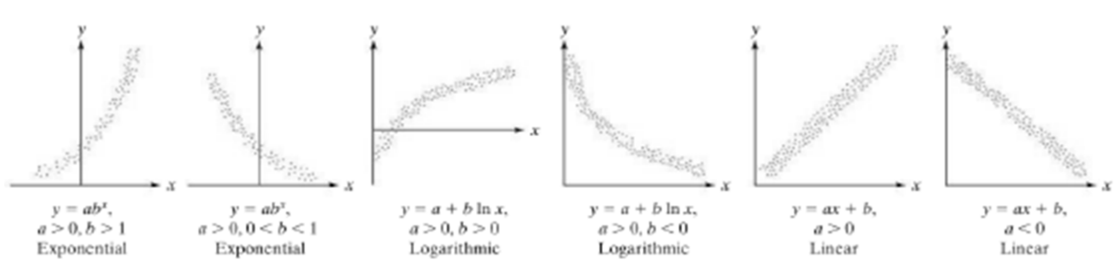
- Logistic Growth Models
  - Logistic growth model: function used to model situations in which growth is limited.
  - Mathematical Model:

$$f(t) = \frac{c}{1+ae^{-bt}} \quad \text{or} \quad A = \frac{c}{1+ae^{-bt}}$$

    - a, b, and c are constants with a > 0 and b > 0
    - y = c is a horizontal asymptote for the graph of the function
    - c represents the limiting size that A can attain (A cannot be bigger than c)

1.8 GROWTH AND DECAY

- Modeling Data
  - Scatter plots can show data that is exponential, logarithmic, and/or linear.



1.8 GROWTH AND DECAY

- A correlation coefficient, r, is a measure of how well the model fits the data.
  - $-1 \leq r \leq 1$; r > 0 indicates a direct relation. r < 0 indicates an inverse relation.
  - The closer to 1 or -1 r is, the better a model fits the data.

1.8 GROWTH AND DECAY

- Expressing an Exponential Model in Base e
  - Recall that $y = ab^x$ is equivalent to $y = ae^{(\ln b)\,x}$
  - Re-writing the equation in this shows the rate of growth or decay.