# Unit 3: Polynomials and Factoring

## 3.1. Add/Subtract Polynomials (M2 11.1)

- Review:
  - Monomial – real number, variable, or product of number and variable
    - Ex: 18 $4x^5$ $y$ $a^2bc^4$
    - Degree of a Monomial – sum of a monomial's variables' exponents
      - constants have a degree of 0
      - Zero has no degree
  - Polynomial – a monomial or sum of monomials
    - Ex: $3x^4$ $5x^2 - 7x$ $2a + 3b + 4c$
    - Standard form of polynomial – ordered from greatest degree to least degree
    - Degree of a polynomial – (in one variable) is the same as the degree of the monomial with the greatest exponent.
- Adding and Subtracting Monomials
  - You can only combine like terms (same variables and degree) with addition and subtraction
  - Add/subtract the coefficients; the variables and exponents remain the same.
  - Examples:
    - $3x^2 + 5x^2 =$ $8x^2$ degree : 2
    - $4x^3y - x^3y =$ $3x^3y$ degree : 4
    - $8w^2x + w^2x =$ $9w^2x$ degree : 3
    - $5bc^4 - 13bc^4 =$ $-8bc^4$ degree: 5
- Classifying Polynomials
  - A polynomial can be named based on its degree, or the number of monomials it contains.

| *Polynomial* | *Degree* | *Name Using Degree* | *Number of Terms* | *Name Using Number of Terms* |
|---|---|---|---|---|
| 6 | 0 | constant | 1 | monomial |
| $5x + 9$ | 1 | linear | 2 | binomial |
| $4x^2 + 7x + 3$ | 2 | quadratic | 3 | trinomial |
| $2x^3$ | 3 | cubic | 1 | monomial |
| $8x^4 - 2x^3 + 3x$ | 4 | Fourth degree | 3 | trinomial |

- Adding Polynomials
  - Add polynomials by combining like terms.
    - Ex: A researcher studied the number of overnight stays in U.S. National Park Service campgrounds and in the backcountry of the national park system over a 5-yr period. The researcher modeled the results, in thousands, with the following polynomials.

      Campgrounds: $-7.1x^2 - 180x + 5800$

      Backcountry: $21x^2 - 140x + 1900$

      In each polynomial, x = 0 corresponds to the first year in the 5-yr period. What polynomial models the total number of overnight stays in both campgrounds and backcountry?

Method 1:

$$\begin{array}{r} -7.1x^2 - 180x + 5800 \\ +\ (21x^2 - 140x + 1900) \\ \hline \boxed{13.9x^2 - 320x + 7700} \end{array}$$

Method 2:

$$(-7.1x^2 + 21x^2) + (-180x - 140x) + (5800 + 1900)$$

$$= \boxed{13.9x^2 - 320x + 7700}$$

  - Simplify $(5x^2 + 3) + (15x^2 + 2)$

$$\begin{array}{r} 5x^2 + 3 \\ +\ (15x^2 + 2) \\ \hline \boxed{20x^2 + 5} \end{array}$$

- Subtracting Polynomials
  - The opposite of addition
  - Distribute the negative into the second polynomial then add the coefficients of like terms.
    - Ex: Simplify $(x^3 - 3x^2 + 5x) - (7x^3 + 5x^2 - 12)$

      Method 1:

$$\begin{array}{r} x^3 - 3x^2 + 5x + 0 \\ +(-7x^3 - 5x^2 - 0x + 12) \\ \hline \boxed{-6x^3 - 8x^2 + 5x + 12} \end{array}$$

Method 2: $(x^3 - 3x^2 + 5x) - (7x^3 + 5x^2 - 12)$

$$(x^3 - 7x^3) + (-3x^2 - 5x^2) + 5x + 12$$

$$\boxed{-6x^3 - 8x^2 + 5x + 12}$$

    - Ex: Simplify $(-6w^4 + w^2) - (-2w^3 + 4w^2 - w)$

$$-6w^4 + 2w^3 + (w^2 + 4w^2) + w$$

$$\boxed{-6w^4 + 2w^3 + 5w^2 + w}$$

## 3.2. Multiply/Factoring Polynomials (M2 11.2)

- Multiply a Monomial and a Trinomial
  - You can use the distributive property to multiply a monomial by a polynomial.
    - Ex: $2x(3x+1)$

$= 2x(3x) + 2x(1)$

$= \boxed{6x^2 + 2x}$

    - Ex: $-x^3(9x^4 - 2x^3 + 7)$

$= -x^3(9x^4) - x^3(-2x^3) - x^3(7)$

$= \boxed{-9x^7 + 2x^6 - 7x^3}$

- Finding the Greatest Common Factor
  - Factoring reverses the multiplication process.
  - When factoring a monomial from a polynomial, the first step is to find the greatest common factor (GCF) of the polynomial's terms.
    - Ex: What is the GCF of the terms of $5x^3 + 25x^2 + 45x$?

$5x^3: 5 \cdot x \cdot x \cdot x$

$25x^2: 5 \cdot 5 \cdot x \cdot x$

$45x: 3 \cdot 3 \cdot 5 \cdot x$

$\boxed{GCF = 5x}$

    - Ex: Find the GCF of the terms of the polynomial $45b + 27$.

$45b: 3 \cdot 3 \cdot 5 \cdot b$

$27: 3 \cdot 3 \cdot 3$

$\boxed{GCF = 9}$

- Factoring Out a Monomial
  - Once you find the GCF of a polynomial's terms, you can factor it out of the polynomial.
    - Ex: What is the factored form of $4x^5 - 24x^3 + 8x$?

$4x^5: 2 \cdot 2 \cdot x \cdot x \cdot x \cdot x \cdot x$

$-24x^3: -1 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot x$

$8x: 2 \cdot 2 \cdot 2 \cdot x$

$GCF = 4x$

$= 4x(x^4) + 4x(-6x^2) + 4x(2)$

$= \boxed{4x(x^4 - 6x^2 + 2)}$

- Ex: Factor the polynomial $g^4 + 24g^3 + 12g^2 + 4g$.

$g^4 : g \cdot g \cdot g \cdot g$

$24g^3 : 24 \cdot g \cdot g \cdot g$

$12g^2 : 12 \cdot g \cdot g$

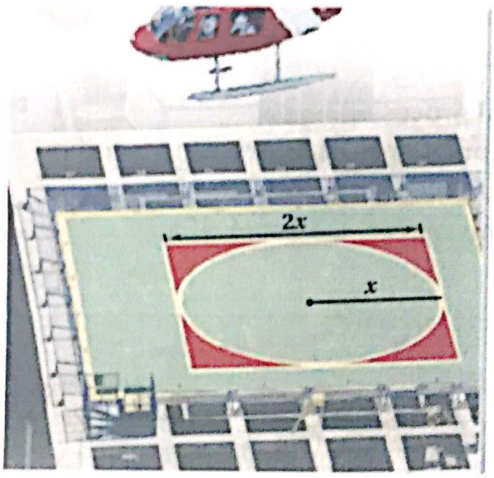
$4g : 4 \cdot g$

$GCF = g$

$g(g^3) + g(24g^2) + g(12g) + g(4)$

$= \boxed{g(g^3 + 24g^2 + 12g + 4)}$

- Factoring a Polynomial Model
  - Ex: A Helicopter landing pad, or helipad, is sometimes parked with a circle inside a square so that it is visible from the air. What is the area of the shaded region of the helipad? Write your answered in factored form.




$A_{square} = 2x(2x)$
$= 4x^2$

$A_{circle} = \pi x^2$

$A_{shade} = 4x^2 - \pi x^2$

$\boxed{A_{shade} = x^2(4 - \pi)}$

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3.3. **Multiplying Binomials & Special Cases (M2 11.3-4)**

- Using the Distributive Property
  - One polynomial can be distributed into another.
  - Ex: What is a simpler form of $(2x+4)(3x-7)$?

$3x(2x+4) - 7(2x+4)$

$6x^2 + 12x - 14x - 28$

$\boxed{6x^2 - 2x - 28}$

  - Ex: Simplify $(m+6)(m-7)$.

$m(m+6) - 7(m+6)$

$m^2 + 6m - 7m - 42$

$\boxed{m^2 - m - 42}$

- Using FOIL
  - FOIL stands for First Outer Inner Last, showing the order to multiply the terms.
  - After terms have been multiplied, combine like terms.
  - This method is only useful when multiplying two binomials.
  - Ex: What is a simpler form of $(5x-3)(2x+1)$?

First Outer Inner Last

$5x(2x) + 5x(1) + (-3)(2x) + (-3)(1)$

$10x^2 + 5x - 6x - 3$

$\boxed{10x^2 - x - 3}$

  - Ex: Simplify $(k-6)(k+8)$.

$K(K) + K(8) + (-6)(K) + (-6)(8)$

$K^2 + 8K - 6K - 48$

$\boxed{K^2 - 2K - 48}$

- Applying Multiplication of Binomials
  - Ex: A cylinder has a height of (x + 4) and a radius of (x + 1). What is the polynomial in standard form that best describes the total surface area of the cylinder? $SA = 2\pi r^2 + 2\pi rh$

$SA = 2\pi(x+1)(x+1) + 2\pi(x+1)(x+4)$

$= 2\pi(x^2+2x+1) + 2\pi(x^2+5x+4)$

$= 2\pi x^2 + 4\pi x + 2\pi + 2\pi x^2 + 10\pi x + 4\pi$

$= \boxed{4\pi x^2 + 14\pi x + 6\pi}$

- Multiplying a Trinomial and a Binomial
  - The vertical method for multiplication works best when multiplying a trinomial and a binomial.

- Ex: What is a simpler form of $(3x^2 + x - 5)(2x - 7)$?

$$\begin{array}{r} 3x^2 + x - 5 \\ 2x - 7 \\ \hline -21x^2 - 7x + 35 \\ 6x^3 + 2x^2 - 10x \quad \\ \hline \boxed{6x^3 - 19x^2 - 17x + 35} \end{array}$$

- Ex: Simplify $(2g + 7)(3g^2 - 5g + 2)$.

$$\begin{array}{r} 3g^2 - 5g + 2 \\ 2g + 7 \\ \hline 21g^2 - 35g + 14 \\ 6g^3 - 10g^2 + 4g \quad \\ \hline \boxed{6g^3 + 11g^2 - 31g + 14} \end{array}$$

- Squaring a Binomial
  - Special rules apply to squaring binomials.
  - $(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$
  - $(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$
  - The square of a binomial is the square of the first term plus twice the product of the two terms plus the square of the last term.
    - Ex: Simplify the following products:

      $(x + 8)^2$

      $= x^2 + 2(8)x + 8^2$

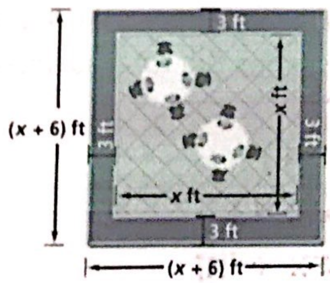
      $= \boxed{x^2 + 16x + 64}$

      $(2m - 3)^2$

      $= (2m)^2 + 2(-3)(2m) + (-3)^2$

      $= \boxed{4m^2 - 12m + 9}$

- Applying Squares of Binomials
  - Ex: A square outdoor patio is surrounded by a brick walkway as shown. What is the area of the walkway?




$(x+6)^2 - (x)^2$

$= x^2 + 2(6)x + 6^2 - x^2$

$= x^2 + 12x + 36 - x^2$

$= \boxed{12x + 36}$

- Finding the Product of a Sum and Difference
  - The product of the sum and difference of the same two terms also produces a pattern.
  - $(a + b)(a - b) = a^2 - b^2$
  - Ex: What is a simpler form of $(x^3 + 8)(x^3 - 8)$?

    $= (x^3)^2 - (8)^2$

    $= \boxed{x^6 - 64}$

## 3.4. Factoring $x^2 + bx + c$ (M2 11.5)

- Factoring $x^2 + bx + c$ where b > 0, c > 0
  - Ex: $(x+3)(x+7) = x^2 + 10x + 21$
    - The coefficient of the trinomial's $x^2$ term is 1.
    - The coefficient of the trinomial's x term, 10, is the sum of 3 and 7 from the binomials.
    - The constant, 21, is the product of 3 and 7 from the binomials.
  - Ex: What is the factored form of $x^2 + 8x + 15$?

| Factors of 15 | Sum of Factors |
|---|---|
| 1 & 15 | 16 |
| 3 & 5 | 8 ✓ |

$\boxed{(x+3)(x+5)}$

  - Ex: Factor $r^2 + 11r + 24$.

| Factors of 15 | Sum of Factors |
|---|---|
| 1 & 24 | 25 |
| 2 & 12 | 14 |
| 3 & 8 | 11 ✓ |
| 4 & 6 | 10 |

$\boxed{(r+3)(r+8)}$

- Factoring $x^2 + bx + c$ where b < 0, c > 0
  - If the coefficient of x is negative, and the coefficient is positive, you will need to look at the negative factors of c.
    - Ex: What is the factored form of $x^2 - 11x + 24$?

| Fac. 24 | Sum Fac. |
|---|---|
| -1 & -24 | -25 |
| -2 & -12 | -14 |
| -3 & -8 | -11 ✓ |
| -4 & -6 | -10 |

$\boxed{(x-3)(x-8)}$

    - Ex: Factor $y^2 - 6y + 8$.

| Fac. 8 | Sum Fac. |
|---|---|
| -1 & -8 | -9 |
| -2 & -4 | -6 ✓ |

$\boxed{(y-2)(y-4)}$

- Factoring $x^2 + bx + c$ where c < 0
  - When you factor trinomials with a negative constant term, you need to inspect pairs of positive and negative factors of c.
    - Ex: What is the factored form of $x^2 + 2x - 15$?

| Fac. -15 | Sum Fac. |
|---|---|
| -1 & 15 | 14 |
| 1 & -15 | -14 |
| -3 & 5 | 2 ✓ |
| 3 & -5 | -2 |

$\boxed{(x-3)(x+5)}$

- Ex: Factor $c^2 - 4c - 21$.

| Fac -21 | | Sum Fac. |
|---|---|---|
| -1 | 21 | 20 |
| 1 | -21 | -20 |
| -3 | 7 | 4 |
| 3 | -7 | -4 ✓ |

$\boxed{(c+3)(c-7)}$

- Applying Factoring Trinomials
  - The area of a rectangle is given by the trinomial $x^2 - 2x - 35$. What are the possible dimensions of the rectangle? Use factoring.

| Fac -35 | | sum Fac |
|---|---|---|
| 1 | -35 | -34 |
| -1 | 35 | 34 |
| 5 | -7 | -2 ✓ |
| -5 | 7 | 2 |

$\boxed{(x+5)(x-7)}$

- Factoring a Trinomial with Two Variables
  - Trinomials with more than one variable can also be factored.
  - Ex: $(p + 9q)(p + 7q) = p^2 + 16pq + 63q^2$
    - The trinomial may be factorable if:
      - the first term includes the square of one variable,
      - the middle term includes both variables, and
      - the last term includes the square of the other variable.
  - Ex: What is the factored form of $x^2 + 6xy - 55y^2$?

| Fac -55 | | Sum Fac |
|---|---|---|
| 1 | -55 | -54 |
| -1 | 55 | 54 |
| 5 | -11 | -6 |
| -5 | 11 | 6 ✓ |

$\boxed{(x-5y)(x+11y)}$

  - Ex: Factor $w^2 - 14wz + 40z^2$.

| Fac 40 | | Sum Fac |
|---|---|---|
| -1 | -40 | -41 |
| -2 | -20 | -22 |
| -4 | -10 | -14 ✓ |

$\boxed{(w-4z)(w-10z)}$

## 3.5. Factoring $ax^2 + bx + c$ (M2 11.6)

- Factoring when ac is positive
  - Consider the trinomial $6x^2 + 23x + 7$. $a=6$ $b=23$ $c=7$ $ac=6(7)=42$
    - To factor it, think of 23x as 2x + 21x (factors of ac).
    - Rewrite the trinomial: $6x^2 + 2x + 21x + 7$
    - Factor out the GCF of each pair of terms:
      $$2x(3x+1) + 7(3x+1)$$
    - Use the Distributive property: $(2x+7)(3x+1)$
  - Ex: What is the factored form of $5x^2 + 11x + 2$?

$a=5$ $b=11$ $c=2$

$ac = 5(2) = 10$

| Fac 10 | Sum Fac |
|---|---|
| 1 10 | 11 ✓ |
| 2 5 | 7 |

$(5x^2 + 10x) + (x+2)$

$5x(x+2) + 1(x+2)$

$\boxed{(x+2)(5x+1)}$

  - Ex: Factor $4n^2 - 8n + 3$.

$a=4$ $b=-8$ $c=3$

$ac = 4(3) = 12$

| Fac 12 | | Sum Fac |
|---|---|---|
| -1 | -12 | -13 |
| -2 | -6 | -8 ✓ |
| -3 | -4 | -7 |

$(4n^2 - 2n) + (-6n + 3)$

$2n(2n-1) - 3(2n-1)$

$\boxed{(2n-3)(2n-1)}$

- Factoring when ac is negative
  - You will need to consider positive and negative factors of ac.
  - Ex: What is the factored form of $3x^2 + 4x - 15$? $ac = 3(-15) = -45$

$a=3$ $b=4$ $c=-15$

| Fac -45 | | Sum Fac. |
|---|---|---|
| -1 | 45 | 44 |
| 1 | -45 | -44 |
| -3 | 15 | 12 |
| 3 | -15 | -12 |
| -5 | 9 | 4 ✓ |
| 5 | -9 | -4 |

$(3x^2 + 9x) + (-5x - 15)$

$3x(x+3) - 5(x+3)$

$\boxed{(3x-5)(x+3)}$

  - Ex: Factor $4w^2 - 5w - 6$. $ac = 4(-6) = -24$

$a=4$ $b=-5$ $c=-6$

| Fac -24 | | Sum Fac |
|---|---|---|
| -1 | 24 | 23 |
| 1 | -24 | -23 |
| -2 | 12 | 10 |
| 2 | -12 | -10 |
| -3 | 8 | 5 |
| 3 | -8 | -5 ✓ |
| -4 | 6 | 2 |
| 4 | -6 | -2 |

$(4w^2 - 8w) + (3w - 6)$

$4w(w-2) + 3(w-2)$

$\boxed{(4w+3)(w-2)}$

- Applying Trinomial Factoring
    - Ex: The area of a rectangle is $2x^2 - 13x - 7$. What are the possible dimensions of the rectangle? Use factoring.

$a=2 \quad b=-13 \quad c=-7$

$ac = 2(-7) = -14$

| Fac -14 | | Sum |
|---|---|---|
| -1 | 14 | 13 |
| 1 | -14 | -13 ✓ |
| -2 | 7 | 5 |
| 2 | -7 | -5 |

$(2x^2 - 14x) + (x - 7)$

$2x(x-7) + 1(x-7)$

$\boxed{(2x+1)(x-7)}$

- Factoring out a Monomial First
    - The GCF of a polynomial must be factored out first before you use any other form of factoring.
    - Ex: What is the factored form of $18x^2 - 33x + 12$?

$a=6 \quad b=-11 \quad c=4$

$3(6x^2 - 11x + 4) \qquad ac = 6(4) = 24$

| Fac 24 | | Sum |
|---|---|---|
| -1 | -24 | -25 |
| -2 | -12 | -14 |
| -3 | -8 | -11 ✓ |
| -4 | -6 | -10 |

$3[(6x^2 - 3x) + (-8x + 4)]$

$3[3x(2x-1) - 4(2x-1)]$

$\boxed{3(3x-4)(2x-1)}$

- Factor $6s^2 + 57s + 72$ completely.

$a=2 \quad b=19 \quad c=24$

$3(2s^2 + 19s + 24) \qquad ac = 2(24) = 48$

| Fac 48 | | Sum |
|---|---|---|
| 1 | 48 | 49 |
| 2 | 24 | 26 |
| 3 | 16 | 19 ✓ |
| 4 | 12 | 16 |
| 6 | 8 | 14 |

$3[(2s^2 + 16s) + (3s + 24)]$

$3[2s(s+8) + 3(s+8)]$

$\boxed{3(2s+3)(s+8)}$

## 3.6. Factoring Special Cases (M2 11.7)

- Factoring a Perfect-Square Trinomial
  - Any trinomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ is a perfect square trinomial because it is the result of squaring a binomial.
  - For every real number a and b:
    - $a^2 + 2ab + b^2 = (a + b)^2$
    - $a^2 - 2ab + b^2 = (a - b)^2$
  - Recognizing a perfect square trinomial:
    - The first and last terms are perfect squares
    - The middle term is twice the product of one factor from the first term and one factor from the second term.
  - Ex: What is the factored form of $x^2 - 12x + 36$?

    $= (x)^2 - 12x + (6)^2$

    $= (x)^2 - 2(6)(x) + (6)^2$

    $= \boxed{(x-6)^2}$
  - Ex: Factor $v^2 + 6v + 9$.

    $= (v)^2 + 6v + (3)^2$

    $= (v)^2 + 2(3)(v) + (3)^2$

    $= \boxed{(v+3)^2}$
- Factoring to Find a Length
  - Ex: Digital images are composed of thousands of tiny pixels rendered as squares. Suppose the area of a pixel is $4x^2 + 20x + 25$. What is the length of one side of the pixel?

    $= (2x)^2 + 20x + (5)^2$

    $= (2x)^2 + 2(2x)(5) + (5)^2$

    $= \boxed{(2x+5)^2} \longrightarrow \boxed{\text{length : } 2x+5}$
- Factoring a Difference of Two Squares
  - Recall that $(a + b)(a - b) = a^2 - b^2$
  - Therefore, you can factor a difference of squares, $a^2 - b^2$ as (a + b)(a – b) .
  - Ex: What is the factored form of $z^2 - 9$?

    $= (z)^2 - (3)^2$

    $= \boxed{(z+3)(z-3)}$
  - Ex: Factor $81m^2 - 225$.

    $= (9m)^2 - (15)^2$

    $= \boxed{(9m+15)(9m-15)}$

- Factoring Out a Common Factor
    - When you factor out a polynomial, sometimes, the expression that remains is a perfect-square trinomial or the difference of two squares.
    - Ex: What is the factored form of $24g^2 - 6$?

$$= 6(4g^2 - 1)$$
$$= 6[(2g)^2 - (1)^2]$$
$$= \boxed{6(2g-1)(2g+1)}$$

    - Ex: Factor $12x^2 + 12x + 3$.

$$= 3(4x^2 + 4x + 1)$$
$$= 3[(2x)^2 + 2(2x)(1) + (1)^2]$$
$$= \boxed{3(2x+1)^2}$$

## 3.7. Factoring by Grouping (M2 11.8)

- Factoring a Cubic Polynomial
  - Some polynomials with a degree > 2 can be factored.
  - When using reverse FOIL (see section 3.5), we grouped terms after replacing bx with factors of ac, and then factored out the GCF from each group.
  - This process is called factoring by grouping.
  - Ex: What is the factored form of $3n^3 - 12n^2 + 2n - 8$?

$= (3n^3 - 12n^2) + (2n - 8)$

$= 3n^2\cancel{(n-4)} + 2\cancel{(n-4)}$

$= \boxed{(n-4)(3n^2+2)}$

  - Ex: Factor $8t^3 + 14t^2 + 20t + 35$.

$= (8t^3 + 14t^2) + (20t + 35)$

$= 2t^2(4t+7) + 5(4t+7)$

$= \boxed{(4t+7)(2t^2+5)}$

- Factoring a Polynomial Completely
  - Before grouping, you may need to factor out the GCF of all the terms.
  - Ex: What is the factored form of $4q^4 - 8q^3 + 12q^2 - 24q$?

$= 4q[q^3 - 2q^2 + 3q - 6]$

$= 4q[(q^3 - 2q^2) + (3q - 6)]$

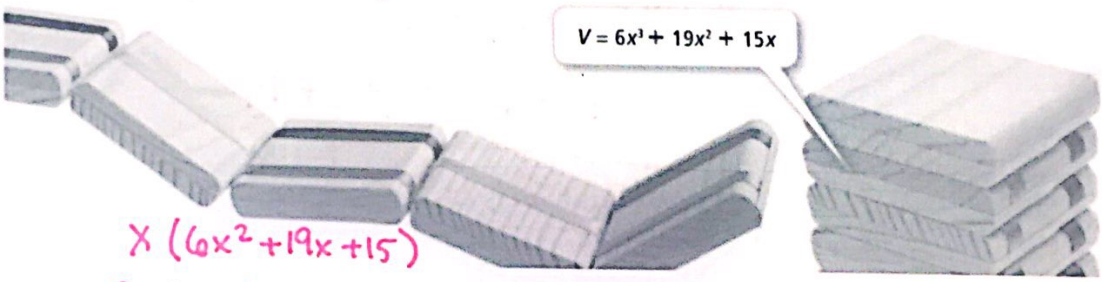
$= 4q[q^2(q-2) + 3(q-2)]$

$= \boxed{4q(q-2)(q^2+3)}$

- Ex: Factor $5g^4 - 5g^3 + 20g^2 - 20g$ completely.

$$= 5g[g^3 - g^2 + 4g - 4]$$
$$= 5g[(g^3 - g^2) + (4g - 4)]$$
$$= 5g[g^2(g-1) + 4(g-1)]$$
$$= \boxed{5g(g-1)(g^2+4)}$$

- Finding the Dimensions of a Rectangular Prism
  - You can sometimes factor to find possible expressions for the length, width, and height of a rectangular prism.
  - Ex: The toy shown below is made of several bars that can fold together to form a rectangular prism or unfold to form a "ladder." What expressions can represent the dimensions of the toy when it is folded up? Use factoring.




$x(6x^2 + 19x + 15)$

$a = 6 \quad b = 19 \quad c = 15 \qquad ac = 90$

| Fac 90 | | Sum |
|---|---|---|
| 1 | 90 | 91 |
| 2 | 45 | 47 |
| 3 | 30 | 33 |
| 6 | 15 | 21 |
| 9 | 10 | 19 ✓ |

$$x[(6x^2 + 9x) + (10x + 15)]$$
$$= x[3x(2x+3) + 5(2x+3)]$$
$$= x(2x+3)(3x+5)$$

dimensions: $x$, $2x+3$, $3x+5$

width, length, height